The Maximal Payoff and Coalition Formation in Coalitional Games*

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Summary: This paper first establishes a new core theorem using the concept of generated payoffs: the TU (transferable utility) core is empty if and only if the maximum of generated payoffs (*mgp*) is greater than the grand coalition's payoff $v(N)$, or if and only if it is irrational to split $v(N)$. It then provides answers to the questions of what payoffs to split, how to split the payoff, what coalitions to form, and how long each of the coalitions will be formed by rational players in coalitional TU games. Finally, it obtains analogous results in coalitional NTU (non-transferable utility) games.

Keywords: Coalition formation, core, maximal payoff, minimum no-blocking payoff.
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1. Introduction

In cooperative game theory with transferable utilities (TU), the previous literature has focused on the question of how to split the grand coalition's payoff. This paper begins by asking a fundamentally different question: Is it always rational to split the grand coalition's payoff? If the answer is no, then in what games is it irrational to split the grand coalitional payoff?

The paper provides conclusive answers by exploring the possibility that players could achieve payoffs higher than the grand coalition's payoff, denoted as $v(N)$. Such exploration leads to the maximum of generated payoffs (*mgp*) for coalitional TU games and leads to the equivalence among three arguments: *i*) it is irrational to split $v(N)$; *ii) mgp* is greater than $v(N)$; and *iii*) the core of the game is empty. In other words, core existence in coalitional TU games can be understood by the rationality of splitting $v(N)$, in addition to the known result that it is balanced (Bondareva [1962], Shapley [1967]) and that its $v(N)$ is greater than the minimum no-blocking payoff (*mnbp*, Zhao [2001]). Because game theory is the study of players' rationality, and because it is irrational to split $v(N)$ in games with an empty core, the equivalence between empty core and the irrationality of splitting $v(N)$ suggests the need to modify previous studies on splitting $v(N)$, which has far-reaching implications for future research in cooperative game theory. In particular, it discourages future research on core enlargements such as the stable set, the bargaining set, and the ε-core, because such non-core splits of $v(N)$ violate players' rationality.

The discovery of new generated payoffs allows us to answer four other (perhaps more important) questions: What payoffs will be split? How will the payoff be split? What coalitions will be formed? and How long will each of the coalitions be formed by rational players in coalitional TU games? Briefly answering these questions (in order), players will split the game's maximal payoff (*mp*), defined as the larger of $v(N)$ and *mgp*; the set of stable splits of *mp* is equal to the core if it is rational to split $v(N)$ (i.e., $mgp \le v(N)$) and equal to the optimal set for *mnbp* if otherwise (i.e., $mgp > v(N)$); players will form coalitions in those minimal balanced collections that generate the game's *mp*; and each coalition in the formed collection will be formed for a length or percentage of time determined by the collection's unique balancing vector.

Finally, the paper obtains analogous results in coalitional non-transferable utilities (NTU) games. Due to the generality of non-transferable utilities, some of the NTU results are weaker than the corresponding TU results. In particular, the irrationality of choosing from the grand coalition's payoff set is only sufficient for an empty NTU core, although the irrationality of splitting $v(N)$ is both necessary and sufficient for an empty TU core.

The rest of the paper is organized as follows. Section 2 reviews the known core results, section 3 studies the generated payoffs and reports a new core theorem, and section 4 studies the maximal payoff and establishes the coalition formation theory. Section 5 obtains analogous results in coalitional NTU games, section 6 concludes, and the appendix provides the proofs.

2. Description of the Problem

This section reviews the concept of the core and its known existence results in coalitional TU games. Let $N = \{1, 2, ..., n\}$ be the set of players, and $\mathcal{N} = 2^N$ be the set of all coalitions. A TU game in coalitional form (or characteristic form), given below,

(1) $$\Gamma = \{N, v(\cdot)\},$$

is a set function $v\colon \mathcal{N} \to \boldsymbol{R}_+$ with $v(\emptyset) = 0$, which specifies a joint payoff $v(S)$ for each coalition $S \in \mathcal{N}$. We use a lowercase v in $v(\cdot)$ to define the above TU game (1), and an uppercase V in $V(\cdot)$ to define coalitional non-transferable utility (NTU) games in section 5.

A payoff vector is any $x = (x_1, \ldots, x_n) \in \mathbf{R}^n_+$, with x_i as player i's payoff for each $i \in N$. Let $X(v(N)) = \{x \in \mathbf{R}^n_+ \mid \Sigma_{i\in N} x_i = v(N)\}$ denote the set of payoff vectors that are splits of $v(N)$, which is often called the *preimputation* or *preimputation space* (see Maschler [1992] for surveys). Given $S \in \mathcal{N}$, a split $x \in X(v(N))$ is unblocked by S if it gives S no less than $v(S)$ (i.e., $\Sigma_{i\in S} x_i \geq v(S)$), and it is in the core (or a core vector) if it is unblocked by all $S \neq N$. Denote the set of all core vectors for the game (1) as

(2) $$c(\Gamma) = \{x \in X(v(N)) \mid \Sigma_{i\in S} x_i \geq v(S) \text{ for all } S \neq N\}.$$

We use a lowercase c in $c(\Gamma)$ to denote TU core and an uppercase C in $C(\Gamma)$ to denote NTU core in section 5. Given the game (1), Bondareva (1962) and Shapley (1969) showed that its core is non-empty if and only if it is balanced. A balanced game is defined below.

Given a collection of coalitions $\mathcal{B} = \{T_1, \ldots, T_k\}$ and a player $i \in N$, the subset of coalitions that include i as a member is $\mathcal{B}(i) = \{T \in \mathcal{B} \mid i \in T\}$. $\mathcal{B}$ is a balanced collection (or balanced) if it has a balancing vector, which is a k-dimensional positive vector $w \in \mathbf{R}^k_{++}$ such that $\Sigma_{T\in\mathcal{B}(i)} w_T = 1$ for each $i \in N$. A balanced collection can be interpreted as a balanced assignment for assigning n students into k ($1 < k < n$) Internet chat rooms (i.e., coalitions or discussion groups). Suppose that each student has one unit of total connection time (= 100 minutes) and could simultaneously join several chat rooms through several connections (i.e., by simultaneously logging onto several computers).

Define an assignment as a pair $(\mathcal{B}, w)$ of chat rooms and opening times, where $\mathcal{B} = \{T_1, \ldots, T_k\}$ is the set of chat rooms ($T_j \neq \emptyset$, all j), and for each $T \in \mathcal{B}$, $w_T > 0$ is the length (or percentage) of time during which chat room T opens (i.e., it opens for $100 \times w_T$ minutes). The set of chat rooms assigned to each student i is $\mathcal{B}(i) = \{ T \in \mathcal{B} \mid i \in T \}$, so i's total participating time is $\Sigma_{T \in \mathcal{B}(i)} w_T$. Then, $(\mathcal{B}, w)$ is a balanced assignment if $\Sigma_{T \in \mathcal{B}(i)} w_T = 1$ for all i. In words, a balanced collection is a balanced assignment such that the total participation time for each student is precisely 100 minutes. Now, the game (1) is balanced if $\Sigma_{T \in \mathcal{B}} w_T v(T) \leq v(N)$ holds for each balanced $\mathcal{B}$ with balancing vector w.

The equivalence between balancedness and the non-empty core was proved by applying the duality theorem to the following linear programming problem (see Myerson [1991], pp. 432-433, and Kannai [1992], pp. 360-361):[1]

(3) $$Min\ \{\Sigma_{i \in N} x_i \mid x \in \mathbf{R}^n_+;\ \Sigma_{i \in S}\, x_i \geq v(S) \text{ for all } S \neq N, \text{ and } \Sigma_{i \in N} x_i = v(N)\}.$$

The minimum no-blocking payoff (*mnbp*) for game (1) is defined as

(4) $$mnbp = Min\ \{\Sigma_{i \in N} x_i \mid x \in \mathbf{R}^n_+;\ \Sigma_{i \in S}\, x_i \geq v(S) \text{ for all } S \neq N\}.$$

The equivalence between $v(N) \geq mnbp$ and non-empty core (Zhao [2001]) is a refinement of the intuition that the core will be non-empty if $v(N)$ is sufficiently large.[2] One advantage of the *mnbp* method is that it characterizes the core's interior: the core has a non-empty (relative) interior if and only if $mnbp < v(N)$ holds.

Although the minimization problem (4) for *mnbp* differs from (3) only in that the grand coalition's constraint is removed, their duality results have completely different

[1] Alternatively, it can be proved by applying the convex hyperplane separation theorem, which leads to the duality theorem for (3). Because the objective of (3) is the constant $v(N)$, its optimal and feasible sets are identical, which coincides with the core. By the duality theorem, $c(\Gamma) \neq \varnothing$ is equivalent to that the dual problem's objective is bounded by $v(N)$, or that the game is balanced.

[2] To see such intuition, let the vertical axis denote $v(N)$, and fix all $v(S)$, $S \neq N$. Now, start with a large $v(N)$ (so its core is non-empty) and keep reducing $v(N)$. The core will eventually become empty after $v(N)$ falls below a critical value, which is equal to the above *mnbp*.

implications. As readers will see in the next two sections, the duality theorem for (4) not only provides a new argument for core existence, but it also answers four other (perhaps more important) questions: What payoffs will be split? How will the payoff be split? What coalitions will form? and How long will each of these coalitions be formed by rational players in our game (1)?

3. The Maximum of Generated Payoffs and a New Core Theorem

In what games is it irrational to split $v(N)$? Let us begin with an inessential game in which $v(N) < \Sigma_{i\in N} v(i)$.[3] Will rational players split $v(N)$ in this game? The answer is no, because players together are better off by splitting $\Sigma_{i\in N} v(i)$, instead of $v(N)$.

Similarly, rational players will not split $v(N)$ in games in which there is a partition Δ such that $v(N) < gp(\Delta) = \Sigma_{S\in\Delta} v(S)$, where $gp(\Delta) = \Sigma_{S\in\Delta} v(S)$ is the payoff generated by the partition Δ.[4] Moving further along this line of argument, we define the payoff generated by a minimal balanced collection[5] and the *mgp* as below:

***Definition 1**: Given game (1) and a minimal balanced collection $\mathcal{B}$ with its unique balancing vector w, the payoff generated by $\mathcal{B}$ is given by $gp(\mathcal{B}) = \Sigma_{T\in\mathcal{B}} w_T v(T)$, and the maximum of generated payoffs (mgp) is given by*

(5) $\quad mgp = mgp(\Gamma) = Max\ \{gp(\mathcal{B}) \mid \mathcal{B} \in B\}$, *where*

(6) $\quad B = \{\mathcal{B} = \{T_1, \ldots, T_k\} \mid N \notin \mathcal{B},\ \mathcal{B}$ *is a minimal balanced collection*$\}$

denotes the set of all minimal balanced collections.

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[3] We simplify $v(\{i\})$ as $v(i)$, $v(\{1,2\})$ as $v(12)$. Similar simplifications apply to other coalitions.

[4] Such payoffs from a partition have been studied for other purposes. For example, Guesnerie and Oddou (1979) and Sun et al. (2005) studied the c-core or C-stable set, and Zhou (1994) studied his bargaining set.

[5] A minimal balanced collection is a balanced collection such that no proper subcollection is balanced. One can show that a balanced collection is minimal if and only if its balancing vector is unique.

The definition considers only minimal balanced collections, because a non-minimal balanced collection is the union of minimal balanced collections. To ameliorate the conceptual difficulty in understanding how a balanced collection could generate the payoff $gp(\mathcal{B})=\Sigma_{T\in\mathcal{B}}w_T\, v(T)$, consider again the problem of assigning n students into k Internet chat rooms, and treat each $v(T)$ as the payoff per unit of time each chat room T receives from advertisers, which also can be understood as the number of visits that T receives per unit of time. Then, the total payoff generated by a balanced assignment $(\mathcal{B}, w)$ is equal to $gp(\mathcal{B})=\Sigma_{T\in\mathcal{B}}w_T\, v(T)$, which is equal to the sum of individual payoffs under the equal-share rule.[6] The following example illustrates such generated payoffs and the irrationality of splitting $v(N)$ in games with $mgp > v(N)$.

Example 1 **(***Internet Assignment Problem***)***: $n= 3$, $v(1) = v(2)= v(3)=0$, $v(12) = v(23) = v(13) = v(123)=$ \$1000. The five minimal balanced collections (excluding* {*N*}*) are the four partitions and $\mathcal{B}_5 = \{12, 13, 23\}$ with a balancing vector* {*0.5, 0.5, 0.5*}. *By (5), mgp = $gp(\mathcal{B}_5)$ = \$1500. The revenue of opening the grand chat room $N = \{1,2,3\}$ for 100 minutes is \$1000, and the revenue of opening each of the two-member chat rooms for 50 minutes is mgp = \$1500 > v(N) = \$1000. Hence, it is irrational to split v(N) = \$1000 in this game, because they could split mgp = \$1500.*

Readers could treat Example 1 as the voting game after dividing each $v(S)$ by 1000 and could predict that a player will form an alliance with each of the other two players for half of the time. This can be completed through a dynamic or virtual process in which a

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[6] Under the equal-share rule, each student $i\in T$ receives $v(T)/|T|$ per unit of time by participating in chat room T. Because each chat room T in $\mathcal{B}$ is opened for w_T units or percentage of the time, i's payoff from $(\mathcal{B}, w)$ is equal to $v(i,\mathcal{B}) = \Sigma_{T\in\mathcal{B}(i)}w_T\, v(T)/|T|$, and the sum of these payoffs are $\Sigma_{i\in N}\, v(i,\mathcal{B}) = \Sigma_{i\in N}\Sigma_{T\in\mathcal{B}(i)}w_T v(T)/|T| = \Sigma_{T\in\mathcal{B}}w_T\Sigma_{i\in T}\, v(T)/|T| = \Sigma_{T\in\mathcal{B}}w_T v(T) = gp(\mathcal{B})$.

player is able to spend one half of his life before (or after) the game or spend two halves of his life simultaneously. Although imaginative, such a process is consistent with empirical evidence. In China's three-kingdom period (220-280 A.D.), for example, two players (Wei and Wu) lived long before the famous three-kingdom game was played.

Denote the maximal (or optimal) set for the above (5) as B_0 given below:

(7) $$B_0 = B_0(\Gamma) = \{\mathcal{B}\in B \mid gp(\mathcal{B}) = mgp\} = Arg\text{-}Max\{gp(\mathcal{B}) \mid \mathcal{B}\in B\}.$$

For each maximal collection $\mathcal{B}\in B_0$ with its unique balancing vector w, it will generate the game's *mgp* when each $T\in\mathcal{B}$ is formed for w_T units (or percentage) of the time.

Note that computing the above *mgp* is not an easy task for a large n, because the number of minimal balanced collections is much larger than the Bell number (i.e., the number of all partitions).[7] However, as shown in Theorem 1 below, one can obtain *mgp* by solving the simpler minimization problem (4) instead of solving (5), because the two problems are dual to each other.

Theorem 1: *Given game (1), the maximization problem (5) for mgp is dual to the minimization problem (4) for mnbp, so mgp = mnbp holds.*

Theorem 1 is proved in the appendix. Theorem 1 leads directly to three equivalent core theorems given below:

Theorem 2: *Given game (1), let its core, mnbp, and mgp be given in (2), (4), and (5), respectively. Then, $c(\Gamma) \neq \varnothing$ is equivalent to each of the following three claims:*

(i) the game is balanced (Bondareva [1962], Shapley [1967]);

(ii) $mnbp(\Gamma) \leq v(N)$ (Zhao [2001]); *and*

[7] Peleg (1965) provides an algorithm for finding all minimal balanced collections. The Bell number (i.e., the number of all partitions) is the sum of Sterling numbers of the second kind. For $n = 1, 2, \ldots, 11$, their Bell numbers are, respectively: *1, 2, 5, 15, 52, 203, 877, 4140, 21,147, 115,975, 678,570*.

(iii) $v(N) \geq mgp(\Gamma)$.

To summarize, there are now three necessary and sufficient empty-core arguments: the game is unbalanced, $v(N)$ is below *mnbp*, and it is irrational to split $v(N)$. This indicates that previous results for splitting $v(N)$ will be *irrational* whenever the core is empty, and it suggests the need to modify all previous studies on splitting $v(N)$, including the more than 10 chapters on core and values in the handbook of game theory (Aumann and Hart [1992]). In particular, it discourages any future research on core enlargements such as the stable set, the bargaining set, and ε-core, because such splits of $v(N)$ violate players' rationality.

4. The Maximal Payoff and Coalition Formation in Coalitional TU Games

The previous section shows that rational players will not split $v(N)$ in games with an empty core. Then, what payoffs will rational players split in games with an empty core? We propose that they will split the maximal payoff defined below:

Definition 2*: The maximal payoff (mp) for game (1) is given by*

(8) $$mp = mp(\Gamma) = Max\ \{mgp, v(N)\},$$

where $mgp = mgp(\Gamma)$ *is the maximum of generated payoffs given in (5).*

It is straightforward to see that $mp = v(N)$ if $c(\Gamma) \neq \varnothing$, $= mgp > v(N)$ if $c(\Gamma) = \varnothing$. Because it is rational to split $v(N) = mp$ if $c(\Gamma) \neq \varnothing$, and $mgp = mp > v(N)$ if $c(\Gamma) = \varnothing$, rational players will always split a game's maximal payoff given in (8), and this answers the question of what payoffs will be split. As shown in Example 1, our three students will split the game's maximal payoff of mp = \$1500, instead of $v(N)$ = \$1000.

Next, consider the question of how to split the maximal payoff. Let the optimal set for *mnbp* in (4) be denoted as Y given below:

(9) $Y = Y(\Gamma) = Arg\text{-}Min\{\Sigma_{i\in N} x_i \mid x \in \boldsymbol{R}^+_n,\ \Sigma_{i\in S}\, x_i \geq v(S) \text{ for all } S\neq N\}$,

which is the set of splits of *mgp* (i.e., $\Sigma x_i = mnbp = mgp$) that possibly can be blocked only by the grand coalition N, because each $x \in Y(\Gamma)$ satisfies the rationality for all $S\neq N$ and all $\mathcal{B} \in B$. Given $x \in Y(\Gamma)$, its stability falls into the following three cases:

Case 1. $mgp > v(N)$, or $c(\Gamma) = \varnothing$. In this case, x is stable against all deviations, because no coalition S (including N) or any minimal balanced collection $\mathcal{B}$ can block it.

Case 2. $mgp = v(N)$, or $c(\Gamma) \neq \varnothing$ and $Int\ c(\Gamma) = \varnothing$, where $Int\ c(\Gamma)$ is the (relative) interior of the core. In this case, x also is stable against all deviations, because $Y(\Gamma) = c(\Gamma)$.

Case 3. $mgp < v(N)$, or $Int\ c(\Gamma) \neq \varnothing$. In this case, x is clearly unstable because it violates the grand coalition's rationality (i.e., $\Sigma x_i = mgp < v(N)$).

The above discussions indicate that the set of stable splits of *mp* is equal to the optimal set $Y(\Gamma)$ if $mgp = mnbp > v(N)$, and the core if $mgp = mnbp \leq v(N)$.

Finally, consider the question of what coalitions will be formed. Because rational players will split the game's maximal payoff, coalitions formed by rational players will support the maximal payoff. By the above properties of *mp*, rational players will form the grand coalition if $v(N) \geq mgp = mnbp$ and the minimal balanced collections in B_0 in (7) if $v(N) < mgp = mnbp$. The unique balancing vector for the formed minimal collection answers the question of how long will each of these coalitions be formed.

The next theorem summarizes the above answers.

Theorem 3: *Given game (1), let its mgp and mp be given in (5) and (8),* $c^* = c^*(\Gamma) \neq \varnothing$ *denote the set of rational splits of mp, and* $B^* = B^*(\Gamma) \neq \varnothing$ *denote the set of stable collections that will be formed. Then, the following three claims hold:*

(i) rational players will split the maximal payoff $mp = mp(\Gamma)$*;*

(ii) the set of rational splits of mp is given by

$$(10) \qquad c^*(\Gamma) = \begin{cases} c(\Gamma) & \text{if } v(N) = mp(\Gamma); \\ Y(\Gamma) & \text{if } v(N) < mp(\Gamma); \end{cases}$$

where $c(\Gamma)$ and $Y(\Gamma)$ are given respectively in (2) and (9); and

(iii) the set of stable collections of coalitions that will be formed is given by

$$(11) \qquad B^*(\Gamma) = \begin{cases} \{N\} & \text{if } v(N) = mp(\Gamma) > mgp(\Gamma); \\ \{N\} \cup B_0(\Gamma) & \text{if } v(N) = mp(\Gamma) = mgp(\Gamma); \\ B_0(\Gamma) & \text{if } v(N) < mp(\Gamma) = mgp(\Gamma); \end{cases}$$

where $B_0(\Gamma)$ is given in (7); and for every $\mathcal{B} \in B^*(\Gamma)$ with its unique balancing vector w, each coalition $T \in \mathcal{B}$ will be formed for w_T unit (or percentage) of the time.

Observe that $c^*(\Gamma) \neq \varnothing$ always holds, so there always exists a split of the maximal payoff that is unblocked by any coalition or any balanced collection. It might be useful to call $c^*(\Gamma)$ in (10) the new core as compared with the old core $c(\Gamma)$ in (2). In the old core, players split $v(N)$ and only rule out deviations by each coalition, whereas in the new core, players split the maximal payoff and rule out not only deviations by each coalition, but also simultaneous deviations by each minimal balanced collection. In the Internet assignment game of Example 1, our three students will form each of the two-member chat rooms for 50 minutes and each will receive \$500; such a split is stable against all possible deviations.

5. Extension to Conational NTU Games

This section answers the questions of what subset of payoffs from which players will choose, how players choose a payoff vector, what coalitions will form, and how long each of these coalitions will be formed in coalitional NTU games. Due to the generality of non-transferable utilities, some of these NTU results are weaker than the corresponding TU results. In particular, conditions for a non-empty NTU core are only sufficient but not necessary.

A coalitional NTU game, or an NTU game in characteristic form, is defined as

(12) $\Gamma = \{N, V(\cdot)\}$,

which specifies a non-empty set of payoffs, $V(S) \subset \boldsymbol{R}^S$, for each $S \in \mathcal{N}$, where $\boldsymbol{R}^S$ is the Euclidean space whose dimension is the number of players in S and whose coordinates are the players in S. For each $S \in \mathcal{N}$, let the (weakly) efficient set of $V(S)$ be given as

$$\partial V(S) = \{\, y \in V(S) \mid \text{there is no } x \in V(S) \text{ such that } x >> y \},$$

where vector inequalities are defined as below: $x \geq y \Leftrightarrow x_i \geq y_i$, all i; $x > y \Leftrightarrow x \geq y$ and $x \neq y$; and $x >> y \Leftrightarrow x_i > y_i$, all i.

Scarf (1967b) introduced the following two assumptions for (12): *(i)* each $V(S)$ is closed and comprehensive (i.e., $y \in V(S)$, $u \in \boldsymbol{R}^S$ and $u \leq y$ imply $u \in V(S)$); *(ii)* for each S, $\{y \in V(S) | y_i \geq \partial V(i) > 0, \text{ all } i \in S\}$ is non-empty and bounded. It is useful to note that $\partial V(i) = Max\ \{x_i \mid x_i \in V(i)\}$. One can check that part (ii) is satisfied in Example 2 (given after Definition 3 in this section). Under these assumptions, each $\partial V(S)$ is closed, non-empty, and bounded from above.

Given $S \in \mathcal{N}$, a payoff vector $u \in \boldsymbol{R}^n_+$ is blocked by S if there is $y \in V(S)$ such that $y >> u_S$ (i.e., $u_S \in V(S) \backslash \partial V(S)$), or in words, if S can obtain a higher payoff for each of its members than that given by u. A payoff vector $u \in \partial V(N)$ is in the core if it is unblocked by all $S \neq N$, so the core of (12) can be given as

(13) $C(\Gamma) = \{\, u \in \partial V(N) \mid u_S \notin V(S) \backslash \partial V(S), \text{ all } S \neq N \}$.

We now define the concept of a balanced NTU game (Scarf [1967b]) geometrically. For each $S \neq N$, let $\tilde{v}(S) = V(S) \times \boldsymbol{R}^{-S} \subset \boldsymbol{R}^n$ denote the n-dimensional cylinder with $V(S)$, where

$\boldsymbol{R}^{-S} = \prod_{i \in N \setminus S} \boldsymbol{R}^i$. Then, the set of payoffs generated by a minimal balanced $\mathcal{B}$, and the set of generated payoffs can be defined as below:

Definition 3*: Given a minimal balanced $\mathcal{B} \in B$, the payoffs generated by $\mathcal{B}$ and the set of generated payoffs in (12) are given, respectively, as*

$$GP(\mathcal{B}) = \bigcap_{S \in \mathcal{B}} \tilde{v}(S) \subset \boldsymbol{R}^n, \text{ and} \tag{14}$$

$$GP = GP(\Gamma) = \bigcup_{\mathcal{B} \in B} GP(\mathcal{B}), \tag{15}$$

where B is the set of minimal balanced collections (excluding N) given in (6).

Note that (14) becomes $GP(\mathcal{B}) = \prod_{S \in \mathcal{B}} V(S)$ when $\mathcal{B}$ is a partition. Similar to the TU case, (15) covers only minimal balanced collections because non-minimal ones are the unions of minimal ones. Readers are encouraged to visualize the generated payoffs in Example 2 below, whose non-negative parts are illustrated in Figure 1.

Example 2: *$n = 3$, $V(i) = \{x_i \mid x_i \leq 1\}$, $i = 1, 2, 3$; $V(12) = \{(x_1, x_2) \mid (x_1, x_2) \leq (3,2)\}$, $V(23) = \{(x_2, x_3) \mid (x_2, x_3) \leq (2,3)\}$, $V(13) = \{(x_1, x_3) \mid (x_1, x_3) \leq (2,2)\}$, and $V(123) = V(N) = \{x \mid x_1 + x_2 + x_3 \leq 7\}$. For the five minimal balanced collections, $\mathcal{B}_1 = \{1, 2, 3\}$, $\mathcal{B}_2 = \{12, 3\}$, $\mathcal{B}_3 = \{23, 1\}$, $\mathcal{B}_4 = \{13, 2\}$, and $\mathcal{B}_5 = \{12, 13, 23\}$, their generated payoffs are: $GP(\mathcal{B}_1) = \{x \mid x \leq (1,1,1)\}$; $GP(\mathcal{B}_2) = \{x \mid x \leq (3,2,1)\}$; $GP(\mathcal{B}_3) = \{x \mid x \leq (1,2,3)\}$; $GP(\mathcal{B}_4) = \{x \mid x \leq (2,1,2)\}$;* and *$GP(\mathcal{B}_5) = \{x \mid x \leq (2,2,2)\}$.*

Now, the NTU game (12) is balanced if

$$GP(\Gamma) \subset V(N) \tag{16}$$

holds, where $GP(\Gamma)$ is the generated payoffs in (15), or in words, (12) is balanced if for each balanced $\mathcal{B}$, $u \in V(N)$ must hold if $u_S \in V(S)$ for all $S \in \mathcal{B}$. To understand a balanced game geometrically, visualize that one is flying in a jet above the Rocky Mountains, and treat the generated payoffs as peaks of the mountains and $V(N)$ as clouds. Then, a game is balanced if one sees only clouds (i.e., $GP(\Gamma) \subset V(N)$, see Figure 2a) and unbalanced if one sees at least one peak above the clouds (i.e., $GP(\Gamma) \not\subset V(N)$, see Figure 2b).

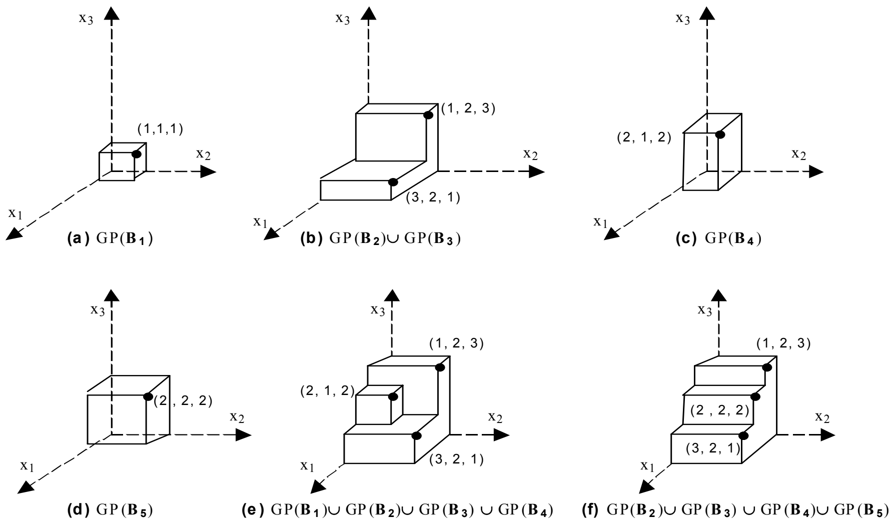

Figure 1. The generated payoffs in Example 2, where $B_1 = \{1, 2, 3\}$, $B_2 = \{12, 3\}$, $B_3 = \{23, 1\}$, $B_4 = \{13, 2\}$, and $B_5 = \{12, 13, 23\}$.

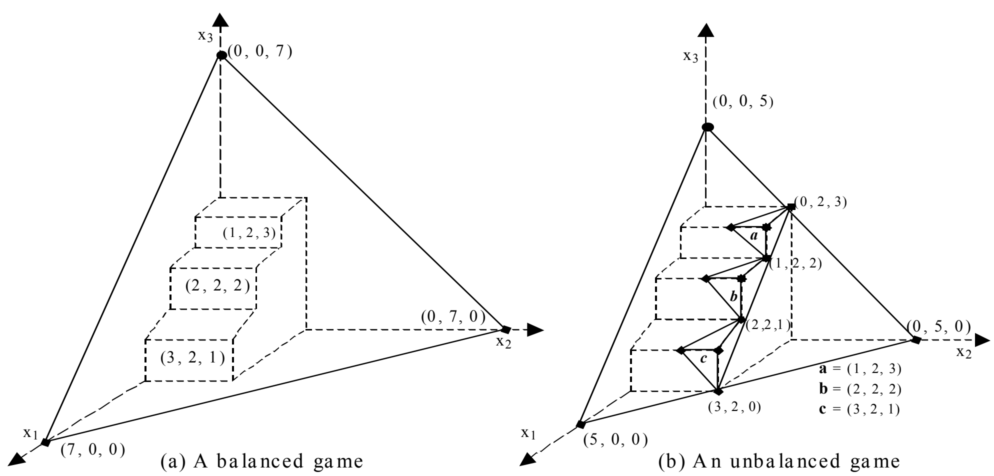

Figure 2. Balanced and unbalanced games.

Figure 2a shows $V(N)$ and the generated payoffs in Example 2. Because one sees only clouds, the game is balanced. Let $V(N)$ be reduced to $V(123) = \{x \mid x_1+x_2+x_3 \leq 5\}$ and all other $V(S)$ remain unchanged. Then, as shown in Figure 2b, because the three peak points a, b, and c are above the clouds or the simplex $X(5) = \{x \in \boldsymbol{R}^3_+ | \Sigma x_i = 5\}$, the game now becomes unbalanced.

Note that the minimal balanced collection $\mathcal{B}_5 = \{12, 13, 23\}$ in Example 2 generates new payoffs that are outside of those generated by the four partitions (i.e., the unit cube next to *[2, 1, 2]* and on the same level; see the difference between [e] and [f] in Figure 1). Similar to Example 1, players could achieve such new generated payoffs in $GP(\mathcal{B}_5)$ by forming each of the two-member coalitions for half of the time. Needless to say, it is the discovery of such new generated payoffs (or the maximum of generated payoff in Example 1) that gives rise to the coalition formation theory introduced in this paper.

Definition 4 below extends the concept of *mnbp* in (4) to minimum no-blocking frontier (*MNBF*), and *mgp* in (5) to (weakly) efficient generated-payoffs (*EGP*). Recall that a payoff vector u is unblocked by S if $u_S \notin V(S) \backslash \partial V(S)$ or if $u \in [V(S) \backslash \partial V(S)]^C \times \boldsymbol{R}^{-S} \subset \boldsymbol{R}^n$, where superscript C denotes the complement of a set. Let

$$UBP = UBP(\Gamma) = \bigcap_{S \neq N} \{[V(S) \backslash \partial V(S)]^C \times \boldsymbol{R}^{-S}\} \subset \boldsymbol{R}^n, \tag{17}$$

denote the set of payoff vectors that are unblocked by all $S \neq N$. Then, the core or (13) becomes $C(\Gamma) = \partial V(N) \cap UBP$, and the concepts of *MNBF* and *EGP* can be defined below.

Definition 4*: Given game (12), let its GP* and *UBP be given in (15) and (17). Let MNBF denote its minimum no-blocking frontier and EGP its efficient generated-payoffs. Then, MNBF and EGP are given by*

$$MNBF = MNBF(\Gamma) = \{\, y \in UBP \mid \exists \text{ no } x \in UBP \text{ such that } x << y\}, \text{ and} \tag{18}$$

$$EGP = EGP(\Gamma) = \partial GP(\Gamma) = \{\, y \in GP \mid \exists \text{ no } x \in GP \text{ such that } x >> y\}. \tag{19}$$

By (18), *MNBF* is the lower boundary or the minimum weakly efficient set of *UBP*.

Any payoff vector on (or above) this boundary is unblocked by all $S \neq N$, this is analogous to the TU result that any solution of (4) given in (9) is unblocked by all $S \neq N$. By (19), EGP is the upper boundary of GP. It will be irrational to choose any $y \in V(N)$ if y is below this boundary; this is analogous to the TU result that it is irrational to split $v(N) < mgp$. Let

(20) $$Z = Z(\Gamma) = MNBF \cap EGP$$

denote the set of unblocked and efficient generated-payoffs. The next theorem shows that $Z \neq \varnothing$ always holds, which is the NTU counterpart of $mgp = mnbp$ in Theorem 1.

Theorem 4: *Given game (12), let* $Z = Z(\Gamma)$ *be given in* (20). *Then,* $Z \neq \varnothing$.

It is straightforward to see that $EGP^* = \{\{1, 2, 3\}; \{2, 2, 2\}; \{3,2,1\}\}$ in Example 2.[8] One can check that none of these three vectors is blocked, so $MNBF \cap EGP \neq \varnothing$ holds in the example.

Theorem 4 is proved by a version of Scarf's closed covering theorem (1967a) due to Zhou (1994). Recall that $EGP \subseteq V(N)$ holds in balanced games. By $MNBF \subset UBP$, $Z = MNBF \cap EGP \neq \varnothing$ leads directly to $C(\Gamma) = \partial V(N) \cap UBP \neq \varnothing$ in balanced games. Hence, our proof of Theorem 4 implies a new proof of Scarf's core theorem.

Now, consider the rationality of choosing a payoff vector from $V(N)$. Similar to the irrationality of splitting $v(N)$ in TU games with $v(N) < mgp$, it will be irrational to choose $u \in V(N)$ if $V(N) \subset GP \backslash \partial GP$ (i.e., if there is $\mathcal{B} \in B$ and $v \in GP(\mathcal{B})$ such that $v >> u$), and rational to choose $u \in V(N)$ if $GP \subseteq V(N)$ (i.e., if the game is balanced). Using our geometric interpretation, it is irrational to choose $u \in V(N)$ if one sees no clouds ($V(N) \subset GP \backslash \partial GP$) and

[8] EGP^* is the efficient set given by $EGP^* = \{y \in GP | \exists$ no $x \in GP$ such that $x > y\} \subseteq EGP$, which is a refinement of the weakly efficient set EGP defined in (19).

rational to choose $u \in V(N)$ if one sees no peaks ($GP \subset V(N)$).

However, unlike in TU games where either $v(N) < mgp$ or $v(N) \geq mgp$ holds, it is possible in NTU games that neither $V(N) \subset GP \backslash \partial GP$ nor $GP \subset V(N)$ holds, or that one sees both clouds and peaks. The existence of such unbalanced games with $V(N) \not\subset GP \backslash \partial GP$ is what makes the following NTU core results weaker than the corresponding TU core results in Theorem 2.

Theorem 5: *Given Γ in (12), let its core, GP and MNBF be given in (13), (15) and (18) respectively. Then, the following three claims hold:*

(i) $C(\Gamma) \neq \varnothing$ *if* $GP \subset v(N)$ (Scarf [1967b]);

(ii) $C(\Gamma) = \varnothing$ *if* $v(N) \subset GP \backslash \partial GP$*; and*

(iii) $C(\Gamma) \neq \varnothing \Leftrightarrow$ *there exists* $x \in \partial v(N)$ *and* $y \in MNBF$ *such that* $x \geq y$.

Comparing Theorem 5 with Theorem 2 leads to the following two differences and one similarity between NTU and TU core results: *i)* balancedness is only a sufficient condition for NTU core existence (Scarf [1967b]), and a necessary and sufficient condition for TU core existence (Bondareva [1962], Shapley [1967]); *ii)* the irrationality of choosing from $V(N)$ is only a sufficient condition for an empty NTU core, whereas the irrationality of splitting $v(N)$ is a necessary and sufficient condition for an empty TU core; and *iii)* "$V(N)$ has a payoff vector on or above MNBF" is a necessary and sufficient condition for NTU core existence, and "$v(N) \geq mnbp$" is a necessary and sufficient condition for TU core existence (Zhao [2001]). As with the TU case, the irrationality of choosing $u \in V(N) \subset GP \backslash \partial GP$ suggests the need to modify previous studies on NTU games with an empty core.

The NTU counterpart of a TU game's maximal payoff in (8) is the following concept of efficient payoffs:

***Definition 6**: The set of efficient payoffs (EP) for our NTU game (12) is given by*

(21) $$EP = EP(\Gamma)=\partial(GP\cup v(N))= \{y\in GP\cup v(N) \mid \exists \text{ no } x\in GP\cup v(N) \text{ with } x>>y\},$$

where $GP = GP(\Gamma)$ is the generated payoff given in (15).

Recall that players in a TU game will always split the maximal payoff defined in (8). Similarly, players in a NTU game will always choose from the set of efficient payoffs defined in (21). This answers the question of what subset of payoffs from which players will choose.

Next, consider the question of how to choose a payoff vector from EP. Let

(22) $$D_0 = D_0(\Gamma)= \{\mathcal{B}\in B \mid GP(\mathcal{B}) \in Z(\Gamma)\}$$

denote the set of minimal-balanced collections that support $Z(\Gamma)$ in (20). For each $\mathcal{B}\in D_0$ with its balancing vector w, it will generate the efficient generated-payoffs in $GP(\mathcal{B}) \in Z(\Gamma)$ when each $T\in\mathcal{B}$ is formed for w_T percentage of the time. As with the TU case, each payoff vector $y\in Z(\Gamma)$ (i.e., $y\in GP(\mathcal{B})$ for some $\mathcal{B}\in D_0$) can possibly be blocked only by the grand coalition N, because the payoff vector y satisfies the rationality for all $S\neq N$ and all $\mathcal{B} \in B$. Hence, $y\in Z(\Gamma)$ is stable if and only if $y\notin V(N)\backslash\partial V(N)$. It will be useful to consider the stability of each $y\in Z(\Gamma)$ in the following three cases.

Case 1. $V(N)\subset GP$. In this case, it is impossible to have $y\in V(N)\backslash\partial V(N)$, so y is stable against deviations by all $S\subseteq N$ and all $\mathcal{B}\in B$.

Case 2. $GP\subset V(N)$. In this case, y is unstable if $y\notin\partial V(N)$ (because it will be blocked by N), and stable if $y\in\partial V(N)$.

Case 3. $V(N)\not\subset GP$ and $GP\not\subset V(N)$. This case is what makes NTU results different from TU results. The stability of y depends on whether $C(\Gamma) \neq \emptyset$. If $C(\Gamma) = \emptyset$, y is stable because N can not block it (otherwise, $C(\Gamma) \neq \varnothing$ holds); if $C(\Gamma) \neq \varnothing$, the stability of y is similar to Case 2: y is unstable if $y \in V(N) \backslash \partial V(N)$, and stable if $y \notin V(N) \backslash \partial V(N)$. Note that $y \in V(N)$ might not hold in Case 3, but it always holds in Case 2.

The above discussions indicate that the set of stable payoffs in EP is $Z(\Gamma)$ in Case 1, $C(\Gamma)$ in Case 2, $C(\Gamma) \cup \{Z(\Gamma) \cap [V(N) \backslash \partial V(N)]^C\}$ in Case 3 with $C(\Gamma) \neq \varnothing$, and $Z(\Gamma)$ in Case 3 with $C(\Gamma) = \varnothing$.

Finally, consider the question of what coalitions will be formed. By earlier arguments, rational players will choose from the set of efficient payoffs in (21), so coalitions formed by rational players shall be either the grand coalition N or the minimal balanced collections from $D_0(\Gamma)$ in (22), which support those efficient payoffs in (21) that are also stable. As with the TU case, the unique balance vector associated with each minimal balanced collection answers the question of how long each of these coalitions will be formed.

The next theorem summarizes the above answers.

Theorem 6: *Given game (12), let $Z(\Gamma)$ and $EP(\Gamma)$ be given in (20) and (21), $C^* = C^*(\Gamma) \neq \varnothing$ denotes the set of stable payoffs from $EP(\Gamma)$, and $D^* = D^*(\Gamma) \neq \varnothing$ denotes the set of minimal balanced collections that will be formed. Then, the following claims hold:*

(i) rational players will choose from the efficient payoffs in $EP(\Gamma)$;

(ii) the set of stable payoff vectors in $EP(\Gamma)$ is given by

$$(23) \qquad C^*(\Gamma) = \begin{cases} C(\Gamma) & \text{if } GP \subset V(N); \\ C(\Gamma) \cup Z(\Gamma)^* & \text{if } V(N) \not\subset GP;\ GP \not\subset V(N);\ C(\Gamma) \neq \varnothing; \\ Z(\Gamma) & \text{if } V(N) \not\subset GP;\ GP \not\subset V(N);\ C(\Gamma) = \varnothing; \text{ or if } V(N) \subset GP; \end{cases}$$

where $Z(\Gamma)^* = Z(\Gamma) \cap [V(N) \backslash \partial V(N)]^C$, and $C(\Gamma)$ is the core given in (13);

(iii) the set of stable collections of coalitions that will be formed is given by

$$
(24) \qquad D^*(\Gamma) = \begin{cases} \{N\} & \text{if } GP \subset V(N); \\ \{N\} \cup D_1(\Gamma) & \text{if } V(N) \not\subset GP;\ GP \not\subset V(N);\ C(\Gamma) \neq \varnothing; \\ D_0(\Gamma) & \text{if } V(N) \not\subset GP;\ GP \not\subset V(N);\ C(\Gamma) = \varnothing;\ \text{or if } V(N) \subset GP; \end{cases}
$$

where $D_1(\Gamma) = \{\mathcal{B} \in D_0(\Gamma) \mid GP(\mathcal{B}) \in Z(\Gamma)^*\}$, $D_0(\Gamma)$ *and* $Z(\Gamma)^*$ *are given in* (22) *and (23).*

Observe that $C^*(\Gamma) \neq \varnothing$ always holds, so there always exists an efficient payoff that is unblocked by any coalition or any balanced collection. Such difference between the new core $C^*(\Gamma)$ in (23) and the old core $C(\Gamma)$ in (13) is the consequence of the possible new generated payoffs. In the old core $C(\Gamma)$, players just choose from $V(N)$ and only rule out deviations by each coalition. In the new core $C^*(\Gamma)$, players also explore the possible higher payoffs generated by minimal balanced collections, choose from the game's efficient payoffs, and rule out not only deviations by each coalition, but also simultaneous deviations by each minimal balanced collection.

6. Conclusion and Discussion

The above analysis revealed the possibility that players in a coalitional game sometimes could achieve better payoffs than the grand coalition's payoffs by forming a minimal balanced collection of coalitions. Our exploration of such opportunity led to the concepts of maximal payoff (*mp*) and efficient payoffs (*EP*) in TU and NTU games, which will be better than the grand coalition's payoff if and only if the core is empty.

In addition to the new core argument, the exploration led to the following four conclusions: *i)* players will achieve the game's *mp* (*EP*) in TU (NTU) games; *ii)* the set of stable payoffs is equal to the core if the core is non-empty and is equal to the optimal set of *mnbp* (the set of unblocked and efficient generated-payoffs) in TU (NTU) games if the core is empty; *iii)* players will form those coalitions in a minimal balanced collection that support the game's *mp* (*EP*) in TU (NTU) games; and *iv)* the unique balancing vector for the minimal balanced collection determines the length (or percentage) of time in which each of the coalitions will be formed.

The irrationality of achieving the grand coalition's payoff in games with an empty core suggests the need to modify previous results for splitting $v(N)$ (or choosing from $V(N)$). Among such a long list of future studies, readers are encouraged to investigate the properties of the following values and refinements of the new core: *i)* modified Shapley value: replacing $v(N)$ with *mp* in Shapley (1953); *ii)* modified nucleolus: replacing $v(N)$ with *mp* in Schmeidler (1969); *iii)* quasi-Shapley value: the vector in $c^*(\Gamma)$ that has the shortest distance between $c^*(\Gamma)$ and the modified Shapley value; *iv)* modified dual nucleolus: the lexicographical maximizer of the ascending excess vector on $c^*(\Gamma)$; and *v)* extensions of *(i-iv)* to coalitional NTU games. Note that *(ii-iv)* are different core selections.

Appendix

Proof of Theorem 1: For each $S \neq N$, let $e_S = (e_1, \ldots, e_n)' \in \boldsymbol{R}_n^+$ be its incidence vector or the column vector such that $e_i = 1$ if $i \in S$ and $e_i = 0$ if $i \notin S$, and $e = e_N = (1, \ldots, 1)'$ be a column vector of ones. Then, the dual problem for the minimization problem (4) is the following maximization problem:

(25) $$Max\ \{\Sigma_{S\neq N} y_S\, v(S) \mid y_S \geq 0 \text{ for all } S\neq N;\text{ and } \Sigma_{S\neq N} y_S\, e_S \leq e\}.$$

We will show that (25) is equivalent to the maximization problem (5). First, we show that the inequality constraints in (25) can be replaced by equation constraints.

Let $Ay \leq e$ and $y \geq 0$ denote the constraints in (25), where y is the $(2^n\text{-}1)$ dimensional vector whose indices are the coalitions, and $A = A_{n\times(2^n-1)} = [e_S \mid S\neq N]$ is the constraint matrix;

let the rows of A be $a_1, \ldots, a_n$. For each feasible y, let $T = T(y) = \{i \mid a_i \cdot y < 1\}$ be the set of loose constraints, so $N \backslash T = \{i \mid a_i \cdot y = 1\}$ is the set of binding constraints.

If $T(y) \neq \emptyset$, let z be defined as: $z_S = y_S + (1 - a_i \cdot y)$ if $S = \{i\}$, for each $i \in T$, and $z_S = y_S$ if $S \neq \{i\}$ for all $i \in T$. One sees that $z > y$ and $T(z) = \emptyset$. Hence, for any y with $T(y) \neq \emptyset$, there exists $z \geq 0$, $Az = e$ such that $\Sigma_{S \neq N} y_S\, v(S) \leq \Sigma_{S \neq N} z_S\, v(S)$. This shows that the feasible set of (25) can be reduced to $\{z \mid z \geq 0,\ Az = e\}$, without affecting the maximum value. So the maximization problem in (25) is equivalent to the following problem:

(26) $$Max\ \{\Sigma_{S \neq N} y_S\, v(S) \mid Ay = e, \text{ and } y \geq 0\}.$$

Note that for each feasible y in (26), $\mathcal{B}(y) = \{S \mid y_S > 0\}$ is a balanced collection. Next, we establish the one-to-one relationship between the extreme points of (26) and the minimal balanced collections. Let y be an extreme point of (26); we will show that $\mathcal{B}(y) = \{S \mid y_S > 0\}$ is a minimal balanced collection.

Assume by way of contradiction that $\mathcal{B}(y)$ is not minimal, then there exists a balanced subcollection $B' \subset \mathcal{B}(y)$ with balancing vector z. Note that $z_S > 0$ implies $y_S > 0$. Therefore, for a small $t > 0$ (e.g., $0 < t \leq ½$, and $t \leq$ Min $\{y_S / (z_S - y_S) \mid$ all S with $y_S < z_S\}$), one has

$$w = y - t(y-z) \geq 0,\ w' = y + t(y-z) \geq 0.$$

$Ay = e$ and $Az = e$ lead to $Aw = e$ and $Aw' = e$. But $y = (w+w')/2$ and $w \neq w'$ contradict the assumption that y is an extreme point. So $\mathcal{B}(y)$ must be minimal.

Now, let $\mathcal{B} = \{T_1, \ldots, T_k\}$ be a minimal balanced collection with a balancing vector z. We will show that z is an extreme point of (26). Assume again by way of contradiction that z is not an extreme point, so there exists $w \neq w'$ such that $z = (w+w')/2$. By $w \geq 0$ and $w' \geq 0$, one has

$$\{S \mid w_S > 0\} \subseteq \mathcal{B} = \{S \mid z_S > 0\}, \text{ and } \{S \mid w_S' > 0\} \subseteq \mathcal{B} = \{S \mid z_S > 0\}.$$

The above two expressions show that both w and w' are balancing vectors for some subcollections of $\mathcal{B}$. Because $\mathcal{B}$ is minimal, one must have $w = w' = z$, which contradicts $w \neq w'$. Therefore, z must be an extreme point of (26).

Finally, by the standard results in linear programming, the maximal value of (26) is achieved among the set of its extreme points, which are equivalent to the set of the minimal balanced collections, so (26) is equivalent to $Max\ \{\Sigma_{S\in\mathcal{B}}\, y_S v(S)\}$, subject to the requirements that $N \notin \mathcal{B}$ and $\mathcal{B}$ is a minimal balanced collection with the balancing vector y. This shows that (25) is equivalent to the maximization problem (5) for *mgp*, which completes the proof for Theorem 1. **Q.E.D**

Proof of Theorem 2: It follows from Theorem 1 and the known results in Bondareva (1962), Shapley(1967), and Zhao (2001). **Q.E.D**

Proof of Theorem 3: The discussion between Definition 2 and the theorem serves as a proof of the theorem. **Q.E.D**

Our proof for Theorem 4 uses the following lemma on open covering of the simplex $\Delta^N = X(1) = \{x \in \boldsymbol{R}_n^+ \mid \Sigma_{i\in N} x_i = 1\}$.

Lemma 2 (Scarf [1967a], Zhou [1994])***:*** *Let $\{C_S\}$, $S \neq N$, be a family of open subsets of Δ^N that satisfy $\Delta^{N\setminus\{i\}} = \{x \in \Delta^N \mid x_i = 0\} \subset C_{\{i\}}$ for all $i \in N$, and $\cup_{S\neq N} C_S = \Delta^N$, then there exists a balanced collection of coalitions $\mathcal{B}$ such that $\cap_{S\in\mathcal{B}} C_S \neq \varnothing$.*

Proof of Theorem 4: Let *UBP* be the set of unblocked payoffs in (17), and *EGP* be the boundary or (weakly) efficient set of the generated payoff in (19). We shall first show that $UBP \cap EGP \neq \varnothing$.

For each coalition $S \neq N$, let $W_S = \{Int\, V(S) \times \mathbf{R}^{-S}\} \cap EGP$ be an open (relatively in EGP) subset of EGP, where $Int\, V(S) = V(S) \backslash \partial V(S)$ is the interior of $V(S)$. For each minimal balanced collection of coalitions $\mathcal{B}$, we claim that

(27) $\cap_{S \in \mathcal{B}} W_S = \varnothing$

holds. If (27) is false, there exists $y \in EGP$ and $y \in Int\, V(S) \times \mathbf{R}^{-S}$ for each $S \in \mathcal{B}$. We can now find a small $t > 0$ such that $y + te \in Int\, V(S) \times \mathbf{R}^{-S}$ for each $S \in \mathcal{B}$, where e is the vector of ones. By the definition of (14) and (15), $y + te \in GP(\mathcal{B}) = \cap_{S \in \mathcal{B}} \{V(S) \times \mathbf{R}^{-S}\} \subset GP$, which contradicts $y \in EGP$. This proves (27).

Now, suppose by way of contradiction that $UBP \cap EGP = \varnothing$. Then, $EGP \subset UBP^C$, where superscript C denotes the complement of a set. The definition of W_S and

$$UBP^C = \{\cap_{S \neq N} \{[V(S) \backslash \partial V(S)]^C \times \mathbf{R}^{-S}\}\}^C = \cup_{S \neq N} \{Int\, V(S) \times \mathbf{R}^{-S}\}$$

together lead to $\cup_{S \neq N} W_S = EGP$, so $\{W_S\}$, $S \neq N$, is an open cover of EGP.

Because the set of generated payoffs is comprehensive and bounded from above, and the origin is in its interior (by $\partial V(i) > 0$, all i), the following mapping from EGP to Δ^N:

$$f: x \rightarrow x / \Sigma x_i,$$

is a homeomorphism. Define $C_S = f(W_S)$ for all $S \subseteq N$, one sees that $\{C_S\}$, $S \neq N$, is an open cover of $\Delta^N = f(EGP)$.

For each $i \in N$, $\partial V(i) > 0$ leads to $EGP \cap \{x \in \mathbf{R}^n \mid x_i = 0\} \subset W_{\{i\}}$, which in turn leads to $\Delta^{N \backslash \{i\}} = \{x \in \Delta^N \mid x_i = 0\} = f(EGP \cap \{x \in \mathbf{R}^n \mid x_i = 0\}) \subset C_{\{i\}} = f(W_{\{i\}})$. Therefore, $\{C_S\}$, $S \neq N$, is an open cover of Δ^N satisfying the conditions of Scarf-Zhou open covering theorem, so there exists a balanced collection of coalitions $\mathcal{B}_0$ such that $\cap_{S \in \mathcal{B}_0} C_S \neq \varnothing$, or that

(28) $\cap_{S \in \mathcal{B}0} W_S \neq \varnothing$,

which contradicts (27). Hence, $UBP \cap EGP \neq \varnothing$.

For each $x \in UBP \cap EGP$, we claim $x \in MNBF$. If this is false, we can find a small $\tau > 0$ such that $x\text{-}\tau e \in UBP$. Let $\mathcal{B} \in B$ be the minimal balanced collection of coalitions such that $x \in GP(\mathcal{B}) = \cap_{S \in \mathcal{B}} \{V(S) \times \mathbf{R}^{-S}\}$. Then, $x\text{-}\tau e \in Int\, V(S) \times \mathbf{R}^{-S}$ for each $S \in \mathcal{B}$, which contradicts $x\text{-}\tau e \in UBP$. Therefore, $MNBF \cap EGP = UBP \cap EGP \neq \varnothing$. **Q.E.D**

Proof of Theorem 5: It follow from the discussions preceding the theorem. **Q.E.D**

Proof of Theorem 6: The conclusions follow from the discussions between Definition 6 and the theorem. **Q.E.D**

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